# §3A Fibre Bundles

This sections adds – as an appendix to §3 – some concepts generalizing line bundles. The aim is to present line bundles within the framework of vector bundles and principal fibre bundles with their associated fibre bundles. The line bundles turn out to be special vector bundles, namely those with 1-dimensional fibres. And vector bundles are in close connection with principal fibre bundles. To uncover these connections we shortly describe the relationship between vector bundles, principal fibre bundles and its associated bundles

## 1. Vector Bundles

A VECTOR BUNDLE over $M$ of rank $r \in \mathbb{N}$, $r \geq 1$, is given by a total space $E$ and a smooth $\pi : E \to M$ such that

$1^\circ$ $E_a := \pi^{-1}(a)$ is an $r$-dimensional vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and

$2^\circ$ $E$ is locally trivial, i.e. there are enough diffeomorphisms

$$\varphi : E_U := \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r$$

($U \subset M$ open) with $pr_1 \circ \varphi = \pi|_{E_U}$ and $\varphi_a = \varphi|_{E_a} : E_a \to \{a\} \times \mathbb{C}^r$ being $\mathbb{K}$-linear.

As before with line bundles a vector bundle $\pi : E \to M$ is again determined by transition functions $(g_{jk})_{j,k}$ with respect to an open cover $(U_j)_{j\in I}$ of $M$ satisfying [C]. But in the case of $r > 1$ the $g_{jk}$ have their values in the group $G = GL(r, \mathbb{K})$ instead of $\mathbb{K}^\times = GL(1, \mathbb{K})$:

$$g_{jk} \in \mathcal{E}(U_{jk}, G).$$

The $g_{jk}$ are given by a collection of trivializations

$$\varphi_j : E_{U_j} = \pi^{-1}(U_j) \to U_j \times \mathbb{K}^r$$

through the condition

$$\varphi_j \circ \varphi_k^{-1}(a, y) = (a, g_{jk}(a).y), \quad (a, y) \in U_{jk} \times \mathbb{C}^r,$$

where $g_{jk}(a).y$ stands for the evaluation of $g_{jk}(a) \in G$ at the vector $y \in \mathbb{C}^r$ (matrix product).

In case of a complex manifold $M$ one also studies holomorphic vector bundles where all occurring maps are holomorphic.

The direct sum, the tensor product, taking the dual, the space of endomorphisms of vector spaces carries over to the vector bundles. Thus we obtain from two vector bundles $E$ and $F$ the bundles

$$E \oplus F, \; E \otimes F, \; E^\vee, \; End(E, F)^{[*]}.$$

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* Übung: Describe transition functions in terms of the transition functions of E and F.

($E^\vee$ is the dual bundle of $E$ with fibre $V^*$ if $V$ is the fibre of $E$.)

## 2. Principal Fibre Bundles (Hauptfaserbündel)

Closely related to vector bundles are the principal fibre bundles with structure group a Lie group $G$. A PRINCIPAL FIBRE BUNDLE with structure group $G$ is a total space $P$ (a manifold) together with a smooth map $\pi : P \to M$ and a differentiable right action $P \times G \xrightarrow{\Psi} P$, $(p,g) \longmapsto pg = \Psi(p,g)$ with $(pg)h = p(gh)$, such that

$1^\circ$ $\pi(pg) = \pi(p)$ for all $(p,g) \in P \times G$ and the action $G \to P_a = \pi^{-1}(a)$, $g \mapsto pg$, is a diffeomorphism for each $p \in P$ with $\pi(p) = a$.

$2^\circ$ there is an open cover $(U_j)$ with local trivalizations

$$\varphi_j : P_{U_j} := \pi^{-1}(U_j) \longrightarrow U_j \times G$$

satisfying $\pi|_{P_{U_j}} = pr_1 \circ \varphi_j$ and

$$\varphi_j(pg) = (a,g) \quad , g \in G, \ a \in U_j$$

for a suitable $p \in \pi^{-1}(a)$ (the $p$ with $\varphi_j(p) = (a,1)$).

These principal fibre bundles are again determined by cocycles $(g_{jk})$, $g_{jk} \in \mathcal{E}(U_{jk}, G)$, defined by the same condition as above:

$$\varphi_j \circ \varphi_k^{-1}(a,y) = (a, g_{jk}(a).y) \quad , (a,y) \in U_{jk} \times G,$$

(see below on page 7 for details).

## 3. Frame Bundle (Reperbündel)

A vector bundle $E \xrightarrow{\pi} M$ induces a principal fibre bundle $P \to M$ with structure group the general linear group $G = GL(r, \mathbb{K})$, the so-called FRAME BUNDLE $R(E)$ of $E \to M$. In our case of a complex line bundle over $M$ the corresponding principle fibre bundle can be described simply by deleting the zero section:

Let $L \xrightarrow[\pi]{} M$ be the line bundle. The map, $a \to 0_a$, where $0_a$ is the zero in $L_a$, defines a section $z$. Let $L^\times := L \setminus z(M)$. Then the restriction of $\pi$ to $L^\times$ defines a principal fibre bundle $L^\times$ over $M$ with the "same" transition functions as $L$.

In the general case the frame bundle $R(E)$ of $E \to M$ is the following: The total space is fiberwise the set $R_a(E)$ of all ordered bases of $E_a$:

$$R(E) = \bigcup_{a \in M} R_a(E), \quad \pi^R(b) = a \quad \text{for } b \in R_a(E)$$

The right action $\Psi : R(E) \times GL(r, \mathbb{K}) \to R(E)$ is the following: If $b = (b_1, \dots, b_r)$ is a basis of $E_a$ then for $g \in GL(r, \mathbb{K})$ one defines

$$(bg)_\varrho := g_\varrho^\sigma b_\sigma \qquad (1 \le \varrho, \sigma \le r).$$

Then $bg = ((bg)_1, \dots, (bg)_r) \in R_a(E)$ and $(bg)h = b(gh)$, and the map $G \to E_a$, $g \mapsto bg$, is bijective.

$R(E)$ obtains its topological and differential structure from the trivializations

$$\varphi : E_U \longrightarrow U \times \mathbb{C}^r.$$

To each $a \in U$ there corresponds a special base $\hat{b}(a)$ of $E_a$ depending on $\varphi$: $\hat{b}(a) = (\varphi^{-1}(a, e_1), \dots, \varphi^{-1}(a, e_r))$, where $e_1, \dots e_r \in \mathbb{C}^r$ is the standard base of $\mathbb{C}^r$. There exists a unique $\hat{\varphi}(b) \in GL(r, \mathbb{C})$ such that

$$b = \hat{b}(a)\hat{\varphi}(b), \quad b \in E_a, \; a \in U.$$

Now, $\varphi^R : R(E)_U \longrightarrow U \times G$, $b \mapsto (\pi(a), \hat{\varphi}(b))$ is bijective with $(\varphi^R)^{-1}(a, g) = \hat{b}(a)g$, $(a,g) \in U \times G$. In particular, $\varphi^R(bh) = (\pi(ah), \hat{\varphi}(bh)) = (\pi(a), \hat{\varphi}(b)h)$

Finally, the topology and the differentiable structure on $R(E)_U$ (and hence on $R(E)$) is defined by requiring all these $\varphi^R$ to be diffeomorphisms.

The construction immediately yields that the transition functions of $E$ are also transition functions of $R(E)$. Thus, up to isomorphisms, $R(E)$ can as well be defined by $(g_{jk})$ directly.

It is evident how to define homomorphisms of vector bundles over $M$, and similarly morphisms of principal fibre bundles: For two such bundles $\pi : P \to M$ and $\pi' : P' \to M$ with structure group $G$ a morphism is a smooth map

$$\psi : P \to P'$$

respecting the projections and the right actions on $P, P'$, i.e. $\pi' \circ \psi = \pi$ and $\psi(pg) = \psi(p)g$, $(p,g) \in P \times G$. (The last property is sometimes called equivariance.)

In particular, we call a principal fibre bundle $P$ over $M$ being TRIVIAL, if there exists a morphism

$$\psi : P \to M \times G$$

which has an inverse $\psi^{-1}$ as a morphism (which means here that $\psi$ is diffeomorphic) and thus establishing an isomorphism of principal fibre bundles.

Moreover, the SECTIONS of the bundles are of considerable interest. In case of a vector bundle $\pi : E \to M$ the set of sections over an open $U \subset M$,

$$\Gamma(U, E) := \{ s \in \mathcal{E}(U, E) \mid \pi \circ s = \mathrm{id}_U \}$$

is an $\mathcal{E}(U)$-module in a natural way, and for a principal fibre bundle $\pi : P \to M$ the set of sections over an open $U \subset M$,

$$\Gamma(U, P) := \{ \sigma \in \mathcal{E}(U, P) \mid \pi \circ \sigma = \mathrm{id}_U \},$$

comes with a natural right $G$-action

$$\Gamma(\mathcal{U},P)\times G \longrightarrow \Gamma(\mathcal{U},P)\,,\ (\sigma,g)\longmapsto \sigma g\,,$$

where $\sigma g(a) := \sigma(a)g$. As an example, the

$$\hat{b} : \mathcal{U} \to R(E)$$

in the construction above is a section.

Note, that a principal fibre bundle $P \overset{\pi}{\to} M$ admitting a section $\sigma \in \Gamma(M,P)$ is already trivial:

$$\varphi^{-1} : M \times G \longrightarrow P\,,\ (a,g)\longmapsto \sigma(a)g\,,$$

defines a trivialization. In this way, every local section $\sigma : \mathcal{U} \to P$ provides a local trivialization:

$$\varphi(b) = (\pi(b), g(b))\,,\ b \in P_{\mathcal{U}}\,,$$

with $b = \sigma(\pi(b))\,g(b)$, $g(b) \in G$ and $g \in \mathcal{E}(P_{\mathcal{U}}, G)$.

## 4. Associated Bundles

How to come back from principal fibre bundles to vector bundles? Take a principal fibre bundle $P \underset{\pi}{\to} M$ with structure group $G$ and with a differentiable left(!) action on a manifold $F$ (the general fibre), denoted

$$G\times F \overset{\lambda}{\longrightarrow} F\,,\ (g,x)\longmapsto gx\,,$$

("action" means $h(gx) = (hg)x$, $g,h\in G, x\in F$), we define $H := P \times_G F := P \times F/\sim$ where the equivalence relation is

$$(p,x) \sim (p',x') :\Longleftrightarrow \exists g \in G : (p',x') = (pg, g^{-1}x)$$

The result is a fibre bundle with structure group $G$. $H$ is called the ASSOCIATED FIBRE BUNDLE.

In the case of $F = \mathbb{C}^r$ and $gx = \rho(g)x$ for a representation $\rho : G \to GL(r,\mathbb{C})$ one obtains a vector bundle $P \times_G \mathbb{C}^r = P \times_\rho \mathbb{C}^r$ of rank $r$, also denoted by $E_\rho$.

Let us compare the transition functions $g_{jk} \in \mathcal{E}(U_{jk}, G)$ of the principal fibre bundle $P \xrightarrow{\pi} M$ with structure group $G$ and the corresponding transition functions of the associated vector bundle $E_\rho$ induced by a representation $\rho : G \to GL(r,\mathbb{C})$:

The result is: $\rho(g_{jk}) \in \mathcal{E}(U_{jk}, GL(r,\mathbb{C}))$ can serve as transition functions of $E_\rho$, and a corresponding result holds for the case of a general associated fibre bundle. We describe this in some detail:

Given local trivializations

$$\varphi_j : P_{U_j} \to U_j \times G \quad , j \in I,$$

for an open cover $(U_j)_{j\in I}$ of $M$, we consider two such trivializations $\varphi_j, \varphi_k$ on $U_{jk} \neq \emptyset$:

$$\begin{tikzcd} U_{jk}\times G \arrow[r, "\varphi_k^{-1}"] \arrow[dr, "pr_1"'] & P_{U_{jk}} \arrow[r, "\varphi_j"] \arrow[d, "\pi"] & U_{jk}\times G \arrow[dl, "pr_1"] \\ & U_{jk} & \end{tikzcd}$$

and observe that the composition $\varphi_j \circ \varphi_k^{-1}$ has the form

$$(a,g) \longmapsto \varphi_j\circ\varphi_k^{-1}(a,g) = (a, g_{jk}(a,g))$$

where $g_{jk} : U_{jk}\times G \to G$ can be written in the form

$$g_{jk}(a,g) = g_{jk}(a)\, g \quad , \text{ with } g_{jk}(a) := g_{jk}(a,1).$$

This follows from $\varphi_j\circ\varphi_k^{-1}(a, gh) = \varphi_j\circ\varphi_k^{-1}(a,g)\, h$, which implies $g_{jk}(a,gh) = g_{jk}(a,g)h$.

<u>PROPOSITION</u>: What we will prove is the following:
Let $G\times F \xrightarrow{\lambda} F$ be a left action on a manifold $F$ in the form of the induced map $\rho : G \to \mathrm{Diff}(M)$ (where $\mathrm{Dif}(M)$ is the group of diffeomorphisms $F\to F$) given by

$$\rho(g)(x) := \lambda(g,f).$$

Then the associated bundle $P\times_\rho F$ has as suitable transition functions $g^F_{jk}$ the functions $\rho(g_{jk}) : U_{jk} \to \mathrm{Diff}(F)$.

□ <u>Proof</u>. In order to show this result we compare – in the situation of a trivialization $\varphi : P_U \to U\times G$ of $P$ over an open $\mathcal{U} \subset M$ - the quotients

$$P_{\mathcal{U}} \times F \longrightarrow H_{\mathcal{U}} = P_{\mathcal{U}} \times_{\varrho} F \text{ and } (\mathcal{U} \times G) \times F \xrightarrow{\eta^F} (\mathcal{U} \times G) \times_{\varrho} F.$$

Because of

$$\eta^F((a,h),x) := [((a,h),x)] = \{((a,hg), \varrho(g^{-1})x) : g \in G\},$$

for $(a,h) \in \mathcal{U} \times P$ and $x \in F$ this class has a unique representative of the form $((a,1), \varrho(h)x) \in (\mathcal{U} \times G) \times F$. We identify $(\mathcal{U} \times G) \times_{\varrho} F$ with $\mathcal{U} \times F$ and note that the quotient map now is

$$(\mathcal{U} \times G) \times F \xrightarrow{\eta^F} \mathcal{U} \times F \,, \quad ((a,h),x) \longmapsto (a, \varrho(h)x).$$

As a result, a suitable trivialization of $H_{\mathcal{U}}$ is

$$H_{\mathcal{U}} \xrightarrow{\varphi^F} \mathcal{U} \times F \,, \quad [(p,x)] \longmapsto (\pi(p), \varrho(\tilde{\varphi}(p)).x),$$

if $\varphi(p) = (\pi(p), \tilde{\varphi}(p))$, $\tilde{\varphi}(p) \in G$, $p \in P_{\mathcal{U}}$.

Now, we conclude

$$\varphi_j^F \circ (\varphi_k^F)^{-1}(a,x) = \varphi_j^F([(\varphi_k^{-1}(a,1), x)]) = (a, \varrho(\varphi_j \circ \varphi_k^{-1}(a,1)).x),$$

and this equality yields the desired result

$$g_{ij}^F = \varrho(g_{ij}) : \mathcal{U}_{ij} \longrightarrow \mathrm{Diff}(F)$$

resp. $$\varrho(g_{ij}) : \mathcal{U}_{ij} \longrightarrow GL(r, \mathbb{C})$$

in the case $\varrho : G \to GL(r, \mathbb{C})$. $\square$

We finally present an example of the process of creating vector bundles as associated fibre bundles induced by a representation – an example which we already know:

<u>EXAMPLE</u>: M is the projective space $\mathbb{P}_n(\mathbb{C}) = M$ of complex dimension $n$. On $M$ we defined the dual $H = T^\vee$ of the tautological line bundle $T \to M$. $H$ is determined by the transition functions (§2 p. 13)

$$g_{jk}(z_0 : \dots : z_n) = \frac{z_k}{z_j}, \quad (z_0 : \dots : z_n) \in U_{jk} = U_j \cap U_k .$$

We have the corresponding principal fibre bundle $H^\times \to M$ with structure group $\mathbb{C}^\times = GL(1,\mathbb{C})$, – the frame bundle –, which is determined by the same transition functions $g_{jk}$.

Now, to each representation $\rho_m : \mathbb{C}^\times \to GL(1,\mathbb{C})$, $z \mapsto z^m$, we obtain the associated vector bundle $E_{\rho_m}$ which is a line bundle with transition functions $\rho_m \circ g_{jk} = {g_{jk}}^m = \left(\frac{z_k}{z_j}\right)^m$ according to our proposition. Hence, $E_{\rho_m}$ is equivalent to our previously defined line bundle $H(m) = H^{\otimes m}$; cf. the discussion at the end of the long example (3.6).